

Quantum metric fluctuations and Hawking radiation

R. Parentani

Laboratoire de Mathématiques et Physique Théorique, CNRS UPRES A 6083, Université de Tours, 37200 Tours, France

In this Letter we study the gravitational interactions between outgoing configurations giving rise to Hawking radiation and in-falling configurations. When the latter are in their ground state, the near horizon interactions lead to collective effects which express themselves as metric fluctuations and which induce dissipation, as in Brownian motion. This dissipation prevents the appearance of trans-Planckian frequencies and leads to a description of Hawking radiation which is very similar to that obtained from sound propagation in condensed matter models.

PACS number(s): 04.70.Dy, 04.60.-m, 05.40.-a

In his original derivation [1], Hawking assumed that gravity can be treated classically, i.e. the metric was determined by the energy of the collapsing star but was unaffected by the quantum processes under examination. In this approximation, the radiation field satisfies a linear equation (in the absence of matter interactions) and the in-falling and outgoing field configurations are completely uncorrelated near the black hole horizon. In fact the pairs of quanta generated by its formation are composed of two outgoing quanta, one of each side of it. The external ones form the asymptotic flux whereas their partners fall towards the singularity at $r = 0$. Upon tracing over these inner configurations one gets an outgoing incoherent flux described by a thermal density matrix.

There is nevertheless a precise relationship between the expectation values of the in-falling and the outgoing energy fluxes. Indeed, the asymptotic thermal flux is accompanied by a negative in-falling flux $\langle T_{vv} \rangle$ which has, on the horizon, exactly the opposite value and which drives black hole evaporation according to

$$\frac{dM}{dv} = \langle T_{vv} \rangle|_{r=r_{\text{horizon}}=2M} \simeq -\frac{1}{M^2}. \quad (1)$$

We work with in Planck units: $c = \hbar = M_{\text{Planck}} = 1$. $v = t + r^*$ is the advanced null time, $r^* = r + 2M \ln(r/2M - 1)$ is the tortoise coordinate and $M(v)$ is the time dependent mass which appears in the Vaidya metric

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2)$$

The important point is that (2) offers a good approximation of the near horizon geometry (i.e. $|r - 2M| \ll 2M$) of an evaporating black hole, see [2,3].

This semi-classical description would be perfectly valid if another feature of black hole physics wasn't present, namely the field configurations giving rise to Hawking quanta possess arbitrary high (trans-Planckian) frequencies near the horizon: When measured by in-falling observers at r , the frequency grows as

$$\omega = \frac{\lambda}{1 - 2M/r} \simeq \frac{\lambda 2M}{r - 2M} \quad (3)$$

where λ is the asymptotic energy of the quantum. This implies that a wave packet centered along the null outgoing geodesic $u = t - r^*$ had a frequency $\omega = \lambda e^{\kappa u}$ when it emerged from the in-falling star. ($\kappa = 1/4M$ is the surface gravity and fixes Hawking temperature $T_H = \kappa/2\pi$.) Unlike processes characterized by a typical energy scale, the relation $\omega = \lambda e^{\kappa u}$ shows that black hole evaporation rests on arbitrary high frequencies.

As emphasized by 't Hooft [4], this implies that the gravitational interactions between these configurations and in-falling quanta cannot be neglected, thereby questioning the validity of the semi-classical description. In questioning this validity, two issues should be distinguished, see Sect. 3.7 in [3]. First, there is the question of the low frequency $O(\kappa)$ changes which can be measured asymptotically, and secondly, that of the high frequency modifications of the near horizon physics. It should also be stressed that thermo-dynamical reasonings indicate that the asymptotic properties, namely thermality governed by κ and stationarity, should be preserved. The difficulty is therefore to conciliate the stability of these properties with the radical change of the near horizon physics which is needed to cure the trans-Planckian problem. Indeed, local interactions near the horizon lead to recoil effects proportional to ω which seem incompatible with the stationarity of the flux [5]. It thus appears inappropriate to treat these interactions by a perturbative approach based on Feynman diagrams.

A different approach is suggested by the analogy with condensed matter physics pointed out by Unruh [6]. He noticed that sound propagation in a moving fluid obeys a d'Alembertian equation which defines an acoustic metric. Therefore, upon neglecting viscosity and non-linear effects, thermally distributed photons should be emitted when the acoustic metric corresponds to that of a collapsing star. (The formation of an acoustic horizon occurs when the fluid velocity reaches the speed of sound.) However contrary to photons, the dispersion relation of phonons is not linear for frequencies (measured in the rest frame of the fluid) higher than a critical ω_c . Nevertheless, when $\omega_c \gg \kappa$, the asymptotic properties of the Hawking flux of phonons are unaffected [7] in spite of the fact that the high frequency spectrum, which was

solicited in Hawking's derivation, is no longer available.

This insensitivity suggests that something similar might apply to black holes and solve the trans-Planckian problem. However it is a priori unclear what can play the role of the microscopic constituents of the fluid which introduce through their interactions the non-trivial dispersion relation and the cut-off ω_c which breaks the low frequency Lorentz invariance [8].

The aim of this letter is to show that the gravitational interactions between the outgoing configurations giving rise to Hawking radiation and in-falling configurations in their ground state lead to collective effects which define an effective dispersion relation for the outgoing modes. These collective effects express themselves in terms of a stochastic ensemble of metric fluctuations. The specification of the vacuum state at early times determines the statistical properties of this ensemble and this in turn fixes both the cut-off ω_c (in terms of κ) and the frame which breaks the 2D Lorentz invariance [9].

For simplicity, we shall consider only s-waves propagating in spherically symmetrical space times. The background metric is taken to be that generated by the collapse of a null spherical shell of mass M_0 which propagate along $v = 0$. Inside the shell, for $v < 0$, the geometry is Minkowski and described by (2) with $M = 0$. Outside, the metric is also static and given by (2) with $M = M_0$.

When studying the propagation of massless s-waves in this background, they fall into two classes according to their support on \mathcal{J}^- , the light-like past infinity. The waves in the first class have support only for $v < 0$ and will be noted ϕ_- . They propagate inward in the flat geometry till $r = 0$ where they bounce off and become outgoing configurations. This first class is itself divided in two: For $v < -4M$, the reflected waves cross the in-falling shell with $r > 2M$ and reach the asymptotic region [10] whereas those for $0 > v > -4M$ cross it with $r < 2M$ and propagate in the trapped region till the singularity. The separating light ray $v_H = -4M$ becomes the future horizon $u = \infty$ after bouncing off at $r = 0$. The configurations which form the second class have support only for $v > 0$ and are noted ϕ_+ . They are always in-falling and cross the horizon towards the singularity.

In the semi-classical derivation of black hole radiation, the configurations for $v < v_H$ give rise to the asymptotic quanta, those for $v_H < v < 0$ to their partners whereas ϕ_+ plays no role in the asymptotic radiation. The correlations between the asymptotic quanta and their partners follow from the fact that, on \mathcal{J}^- and in the vacuum, the rescaled field $\phi = r\chi$ (where χ is the 4D s-wave) satisfies

$$\langle \phi(v)\phi(v') \rangle = \int_0^\infty \frac{d\omega}{4\pi\omega} e^{-i\omega(v-v')} = \frac{1}{4\pi} \ln|v-v'| \quad (4)$$

across v_H . Since this equation is valid for all v, v' there also exist correlations between ϕ_- and ϕ_+ . However, they become negligible for late Hawking quanta since these

emerge from configurations which are characterized by frequencies $\omega = \lambda e^{\kappa u} \gg \kappa$ and which are extremely localized across v_H , see [11]. These two effects follow from the asymptotic ($\kappa u \gg 1$) relation [1] between the value of u of the geodesic which originates from v on \mathcal{J}^-

$$V(u) - v_H \simeq e^{-\kappa u}. \quad (5)$$

It is this exponential which induces both the thermal radiation at temperature $\kappa/2\pi$ and the necessity of considering trans-Planckian frequencies. In the absence of gravitational interactions, it also tells us that ϕ_- and ϕ_+ are effectively two independent fields.

Our aim is now to describe how the gravitational interactions between ϕ_- and ϕ_+ modify the semi-classical description of Hawking radiation. The generating functional of the whole system is

$$Z = \int \mathcal{D}\phi_- \mathcal{D}\phi_+ \mathcal{D}h e^{i[S_{g+h}^{(-)} + S_{g+h}^{(+)} + S_{h,g}]}. \quad (6)$$

In this equation, h is the linear change of the metric with respect to the background g discussed above and $S_{h,g}$ is the action of h obtained from the Einstein-Hilbert action. $S_{g+h}^{(-)}$ and $S_{g+h}^{(+)}$ are the actions of ϕ_- and ϕ_+ propagating in the fluctuating geometry $g+h$.

When imposing that the metric fluctuations be spherically symmetric, h is determined by the matter stress tensor and characterized by ψ and μ , two functions of v and r [12]. In the place of (2), the line element in the fluctuating metric is

$$ds^2 = (1 + \psi) \left[\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr \right] + r^2 d\Omega_2^2 \quad (7)$$

where $M = M_0 + \mu(v, r)$. In this metric, the matter action is

$$S_{g+h} = \int dvdr \left[\partial_v \phi \partial_r \phi - \left(1 - \frac{2M}{r}\right) \frac{(\partial_r \phi)^2}{2} \right]. \quad (8)$$

It is independent of ψ , thereby showing an horizon induced 2D conformal invariance, *c.f.* [9,10].

When only quadratic terms in h are kept in $S_{h,g}$ the functional integration over h gives

$$Z = \int \mathcal{D}\phi_- e^{iS_g^{(-)}} \int \mathcal{D}\phi_+ e^{iS_g^{(+)} + iS_{int}} \quad (9)$$

where the interaction action S_{int} is a quadratic form of the total stress tensor. It thus contains self-interaction terms depending on ϕ_- or ϕ_+ separately. The terms concerning ϕ_- only have been studied in [13] and lead to small effects. Those concerning ϕ_+ are not directly relevant to black hole radiation since the ϕ_+ configurations disappear through the horizon. We shall simply assume that these interactions do not significantly modify (4) for $\omega \simeq \kappa$, a weak condition for large black holes $M \gg M_{Planck}$.

S_{int} also contains cross terms which couple ϕ_- to ϕ_+ . When focusing at the near horizon physics, i.e. in the semi-infinite strip $|r - 2M| \ll 2M$, $v > 0$, the relevant metric fluctuation is given by a mass term μ_+ which is determined by $T_{vv} = (\partial_v \phi_+)^2$

$$\mu_+(v) = \mu(v, r)|_{r=2M} = \int_0^v dv' T_{vv}(v'). \quad (10)$$

Using this approximation, the dominant contribution to S_{int} is given by, see (8),

$$S_{int} = \int_0^\infty dr \int_0^\infty dv \frac{\mu_+(v)}{r} (\partial_r \phi_-)^2. \quad (11)$$

The reason that μ_+ gives the dominant contribution is that it is coupled to $(\partial_r \phi_-)^2 \simeq (\partial_v \phi_-)^2 / (r/2M - 1)^2$ which diverges for $r \rightarrow 2M$. In brief, (11) represents the $\phi_- \phi_+$ interactions mediated by gravity. They have been already considered in [4,14,15]. The novelty of this letter lies in their collective treatment when the state of ϕ_+ is vacuum.

When using (11) in (9) three approximations can be considered. The first one is that adopted by Hawking. It simply consists in putting $\mu_+ = 0$. Then ϕ_- is a free field propagating in the background geometry g and ϕ_+ drops out from all matrix elements built with the operator ϕ_- . In this treatment these matrix elements are characterized by trans-Planckian frequencies when one of the operator approaches the future horizon [11].

The second approximation consists in working with the expectation value of (10). Then the mean metric change $\langle \mu_+(v) \rangle$ is driven by the (properly subtracted [3]) expectation value of T_{vv}

$$\langle T_{vv}(v) \rangle|_{r=2M} = -\frac{\pi}{12} \left(\frac{\kappa}{2\pi} \right)^2. \quad (12)$$

This is minus the usual value of a 2D thermal flux. It is this negative flux which drives the black hole evaporation, see (1). In this mean field approximation, one simply replaces M_0 by $M_0 + \langle \mu_+ \rangle$ in the former treatment. Therefore, the trans-Planckian problem mentioned above stays as such as long as this mass loss is slow compared to κ , i.e. as long as $M(v) \gg M_{Planck}$.

To solve this problem clearly requires to take into account the fluctuating character of the interactions between ϕ_- and ϕ_+ . By proceeding formally, one would perform in (9) the integration over ϕ_+ so as to obtain the influence functional (IF) [17] for the ϕ_- configurations on which we have access asymptotically. However to perform exactly this functional integration is out of reach since the final state of ϕ_+ is correlated to that of ϕ_- . A convenient approximation consists in neglecting the entanglement. This is a common procedure both in QFT where it gives the vacuum contribution, see Chapt. 9 in [17], and in statistical mechanics (e.g. the *polaron*, Chapt. 11) when one neglects the modification of the environment due to the interactions with the system.

In this third approximation, the IF gives rise to a non-local action which is a sum of terms containing $(\partial_r \phi_-)^2$ and kernels given by the Wick contractions of the $\partial_v \phi_+$. The first term is quadratic in $(\partial_r \phi_-)^2$ and the kernel is the (connected) two-point function of T_{vv}

$$\langle T_{vv}(v) T_{vv}(v') \rangle_c = \frac{1}{16\pi^2} \frac{1}{(v - v')^4} \quad (13)$$

evaluated in the vacuum (4). Keeping only this term in the IF is equivalent [16] to work with a stochastic ensemble of metric fluctuations μ_+ which obey, see (10),

$$\begin{aligned} \langle \mu_+(v) \mu_+(v') \rangle &= \frac{1}{96\pi^2} \frac{1}{(v - v')^2} \\ &= \frac{1}{96\pi^2} \int_0^\infty d\omega \omega \cos[\omega(v - v')]. \end{aligned} \quad (14)$$

This equation gives the mean properties of the spherical metric fluctuations driven by ϕ_+ in its vacuum state.

In this quadratic approximation, to exploit the near horizon conformal invariance of (8), one should perform the integration over ϕ_+ after that over ϕ_- [18]. In this eikonal-like treatment, the non-linear effects induced by the metric fluctuations are taken into account through the characteristics of the equation of motion of ϕ_-

$$\left(1 - \frac{2M_0 + 2\mu_+(v)}{r}\right) \partial_r \phi_- = 2\partial_v \phi_-. \quad (15)$$

These are nothing but the outgoing null geodesics $u(v, r)$, the non trivial solutions of $ds^2 = 0$ of (7). The background solution is $u_0 = v - 2r^*$. The first order change induced by μ_+ , $\delta u = u - u_0$, is given by [12]

$$\delta u(v)|_{u_0} = \int_v^\infty dv' \frac{\mu_+(v')}{r(v')|_{u_0} - 2M_0} \quad (16)$$

where $r(v)|_{u_0}$ is obtained by inverting $u_0(v, r) = v - 2r^*$. The integral is dominated by the near horizon region wherein $r(v)|_{u_0} - 2M_0 \simeq 2M_0 e^{\kappa(v-u)}$. This dependence in κv will tame the UV content of the metric fluctuations.

To determine the effects of these fluctuations on outgoing configurations, we analyze the near horizon behavior of asymptotic plane waves representing Hawking quanta. (Notice that it also controls that of matrix elements of ϕ_- such as the Feynman Green function.) In the absence of metric fluctuations $e^{-i\lambda u}$ behaves as

$$e^{-i\lambda u_0(v, r)} = \theta(r - 2M_0) e^{-i\lambda v} (r - 2M_0)^{i\kappa\lambda}. \quad (17)$$

It vanishes for $r < 2M_0$ and possesses an infinite number of oscillations as $r \rightarrow 2M_0$ with increasing momentum $p_r = -i\partial_r$. This is the trans-Planckian problem.

When considering the Feynman Green function obtained from (9), (11) and (13) (and with one operator at v, r and the other on \mathcal{J}^+), its behavior in v, r is determined [12] by the ensemble average waves

$$\langle e^{-i\lambda u(v,r)} \rangle \simeq e^{-i\lambda u_0(v,r)} e^{-\frac{\Lambda^2}{2} \langle \delta u(v) \delta u(v) \rangle}. \quad (18)$$

Using (14) and (16), one obtains

$$\begin{aligned} \langle \delta u(v)|_{u_0} \delta u(v)|_{u_0} \rangle &= \int_0^\infty d\omega \frac{\kappa^2 \omega}{12 \kappa^2 + \omega^2} \frac{1}{(r/2M_0 - 1)^2} \\ &= \bar{\sigma}_\Lambda^2 \frac{1}{(r/2M_0 - 1)^2} \end{aligned} \quad (19)$$

where the mean spread $\bar{\sigma}_\Lambda$ is equal to $\kappa \sqrt{\ln(\Lambda/\kappa)/12}$. We have introduced the UV cut-off Λ to define the integral over ω . Notice that Λ is a Lorentz scalar (since it is the energy of an s-wave in its rest frame) and that its value is hardly relevant since $\bar{\sigma}_{\Lambda=M} = \sqrt{2} \bar{\sigma}_{\Lambda=1}$.

The main result of (19) is that $\bar{\sigma}_\Lambda$ is not proportional to Λ even though $\langle \mu_+^2 \rangle \simeq \Lambda^2$. This is because the high frequencies ($\omega \gg \kappa$) are damped by the integration over v in (16). The frequencies $\omega \simeq \kappa$ dominate the integral.

Since $\langle \delta u \delta u \rangle$ diverges as $r \rightarrow 2M_0$, (18) tells us that the correlations between asymptotic quanta and early configurations, which existed in a given background as shown in (17), are washed out by the metric fluctuations once $r - 2M_0 \simeq \bar{\sigma}_\Lambda \simeq 1/M_0$. The physical reason of this loss of coherence is that the state of ϕ_+ becomes correlated to that of ϕ_- [4,14]. Nevertheless, when tracing over both ϕ_+ and the inner configurations of ϕ_- , the asymptotic properties of Hawking radiation obtain [12].

Finally it should be pointed out that (18) can be viewed as defining a phenomenological dissipation of outgoing waves. In this point of view, as in condensed matter [6,8], one ignores the interactions with the environment and deals only with an effective wave propagation governed by a non-trivial dispersion relation.

In conclusion, even though we have made many simplifying assumptions, we believe that the following results are robust. (A) When propagated backwards in time towards the star's matter, outgoing quanta are scattered by the metric fluctuations induced by the in-falling quantum matter fields in their vacuum state. (B) These interactions grow so strongly near the horizon that the quanta are completely scattered. This prevents a perturbative S -matrix description of these interactions. (C) The in-falling vacuum fluctuations act as a reservoir of modes. This invites to describe these interactions in terms of stochastic metric fluctuations. (D) Even though the spectrum of the latter contains all frequencies (up to a UV cut-off), their impact on outgoing configurations is governed by frequencies $\omega \simeq \kappa$. (E) The stationarity of

vacuum (i.e. the fact that (13) is a function of $v - v'$ only) leads to stationary metric fluctuations (14) and this, combined with the stationarity of the background metric, give a mean spread $\bar{\sigma}_\Lambda$ which is independent of v .

Acknowledgements. I wish to thank Robert Brout for very useful discussions. I also thank the organizers of the 5-th Peyresq meeting where some of the ideas presented here crystallized.

-
- [1] S. W. Hawking, Commun. Math. Phys. **43** (1975) 199.
 - [2] S. Massar, Phys. Rev. D **52** (1995) 5857.
 - [3] R. Brout *et al.*, Phys. Rep. **260** (1995) 329.
 - [4] G. 't Hooft, Nucl. Phys. B **256** (1985) 727 and Int. J. Mod. Phys. A **11** (1996) 4623.
 - [5] R. Parentani, Phys. Rev. D **61** (1999) 027501.
 - [6] W. G. Unruh, Phys. Rev. Lett. **46** (1981) 1351.
 - [7] W. G. Unruh, Phys. Rev. D **51** (1995) 2827.
 - [8] T. Jacobson, Phys. Rev. D **44** (1991) 1731, D **53** (1996) 7082.
 - [9] In the vicinity of a black hole horizon, there is an induced Lorentz invariance in the u, v plane. This can be seen from the fact that the mass term and the contribution of the centrifugal barrier appeared multiplied by $r - 2M$ in the 4D d'Alembertian. Indeed, the near horizon equation reduces to $\partial_u \partial_v \phi = 0$.
 - [10] $\partial_u \partial_v \phi = 0$ is valid for all r only when working in the geometric optic approximation. In the exact d'Alembertian there is a potential around $r = 3M$ which induces partial reflection, a phenomenon irrelevant for our purposes.
 - [11] S. Massar and R. Parentani, Phys. Rev. D **54** (1996) 7444.
 - [12] Barrabès *et al.*, Phys. Rev. D **62** (2000) 044020.
 - [13] P. Kraus and F. Wilczek, Nucl. Phys. B **433** (1995) 403, and S. Massar and R. Parentani, Nucl. Phys. B **575** (2000) 333.
 - [14] Y. Kiem, H. and E. Verlinde, Phys. Rev. D **52** (1995) 7053.
 - [15] A. Casher *et al.*, Nucl. Phys. B **484** (1997) 419.
 - [16] R. Martin and E. Verdaguer, Phys. Rev. D **61** (2000) 124024.
 - [17] R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hills, New-York, 1965.
 - [18] By integrating over ϕ_+ first, one obtains a nice diagrammatic representation for the near horizon interactions between ϕ_- and ϕ_+ in its vacuum. However, this non-linear field description is not easy to handle.